Analytical results of asymmetric exclusion processes with ramps

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We present the analytical results in a simple traffic model describing a single-lane highway with ramps. Both on-ramps and off-ramps are considered. Complete classification of distinct phases is achieved. Exact phase diagrams are derived. In the case of a single ramp (either on-ramp or off-ramp), the bottleneck effect is absent. The traffic conditions of congestion before the ramp and free-flowing after the ramp cannot be realized. In the case of two consecutive ramps, the bottleneck emerges when the on-ramp is placed before the off-ramp and the flow in between the ramps saturates.

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I. INTRODUCTION

With recent attentions on the basic ingredients of traffic modeling, the complexity of transportation can be represented as simple processes of cellular automaton [1,2]. Along this line, the asymmetric simple exclusion process (ASEP) can be taken as the most basic version of traffic models [3,4]. In such a nonequilibrium process, a steady current is maintained in the asymptotic state. The bulk properties are controlled by the conditions imposed on the boundaries. Thus the dynamics can be characterized as the boundary-induced phase transition [5,6]. For a simple roadway modeled as a one-dimensional lattice, the transition from free flow to congestion is determined by the boundary conditions at the two ends. When the ramps are introduced, traffic conditions would be much more complicated. Recently, the effects of an on-ramp have been observed empirically [7–10]. A new intrinsic traffic state has been proposed [11–14]. As the ramps also served as a special kind of boundary, the strong influence to the bulk properties is expected [15]. However, the ramps introduce new dynamics and new parameters into the models. The phase diagram becomes complicated and difficult to classify [16]. In the cases without ramps, the exact results exist for some simple models [17–20]. With ramps, to our knowledge, most of the previous works are focused on the numerical aspects of the models [21–25]. In the following, we would like to explore the analytic properties of the ramp effects. In the case of a single off-ramp, the numerical results reported in Ref. [25] can be reproduced analytically. Besides, the exact results of a single on-ramp and the combining effect of two ramps are also obtained.

The model with an on-ramp is introduced in the next section. The effects of an off-ramp are presented in Sec. III. The combining effects of two ramps are discussed in Sec. IV. Section V is the concluding remarks.

II. ON-RAMP MODEL

ASEP is a simple cellular automaton on one-dimensional lattice. Each site can be either empty or occupied by a particle. The system configurations are updated in discrete time steps by a simple rule: a particle hops forward to the next site provided an empty site is available. The movements of all the particles are synchronized, i.e., the update rule is applied in parallel to all particles. The ASEP describes particles flowing deterministically and unidirectionally from left to right. In the two boundaries, the injection and removal of particles are controlled by two stochastic variables. In the left boundary, a particle is injected with a rate α provided the first site is empty; in the right boundary, a particle is removed with a rate β provided the last site is occupied. The system shows two distinct phases determined by these two rates. For $\alpha < \beta$, particles flow freely. The bulk properties are controlled by the left boundary, i.e., the rate of injection α,

$$\rho = \frac{\alpha}{1+\alpha} = j, \tag{1}$$

where ρ and j are the bulk density and current, respectively. In contrast, for $\alpha > \beta$, the flow is congested. The bulk properties are then controlled by the right boundary, i.e., the rate of removal β,

$$\rho = \frac{1}{1+\beta} = 1 - j. \tag{2}$$

Next, we place an on-ramp on the lattice and a particle is added with a rate γ provided that site is empty. With such a configuration, the whole lattice can be separated effectively into two sublattices. The injection rate γ from the on-ramp can be replaced by an effective removal rate β' for the first part of the lattice and an effective injection rate α' for the second part of the lattice. In the cases of congestion $\alpha > \beta$, the on-ramp would only make the flow more congested. Thus we have

$$\rho_1 = \frac{1}{1+\beta'} = 1 - j_1, \tag{3}$$

$$\rho_2 = \frac{1}{1+\beta} = 1 - j_2, \tag{4}$$

where the subscripts 1 and 2 denote the first and the second parts of the lattice, respectively. Effectively, we have $\alpha' = 1$. The flow balance provides

$$j_1+(1-\rho_2)\gamma=j_2, \tag{5}$$

which can be used to solve β' analytically,

$$\beta' = \frac{\beta(1-\gamma)}{1+\beta\gamma}, \tag{6}$$

which then implies

$$\rho_1 = \frac{1+\beta\gamma}{1+\beta}. \tag{7}$$

When $\gamma=0$, we have $\rho_1=\rho_2$. As γ increases, ρ_1 increases with ρ_2 kept constant. The injection from the on-ramp will not influence the flow after the on-ramp, while the flow before the on-ramp will be hindered. The more injection from the on-ramp, the more severe the hindrance will be.

In the cases of free flow $\alpha<\beta$, the congested situations can also be expected if γ is large enough. The congested condition $\alpha>\beta'$ can then be used to determine the boundary of the congested phase. Then we have

$$\gamma > \frac{\beta-\alpha}{\beta(1+\alpha)}. \tag{8}$$

On the contrary, the free flow can be resumed if γ is small enough. In such situations, we have

$$\rho_1 = \frac{\alpha}{1+\alpha} = j_1, \tag{9}$$

$$\rho_2 = \frac{\alpha'}{1+\alpha'} = j_2. \tag{10}$$

Similarly, the flow balance shown in Eq. (5) can be used to determine α' analytically,

$$\alpha' = \alpha + \gamma(1+\alpha), \tag{11}$$

which then implies

$$\rho_2 = \frac{\alpha+\gamma(1+\alpha)}{(1+\alpha)(1+\gamma)}. \tag{12}$$

The effective removal rate β' can also be determined accordingly,

$$\beta' = \frac{\alpha(1+\gamma)}{\alpha+\gamma(1+\alpha)}, \tag{13}$$

which is irrelevant to the bulk properties. Again, when $\gamma=0$, we have $\rho_1=\rho_2$. As γ increases, ρ_2 increases with ρ_1 kept constant. The on-ramp injection will not influence the free flow before the ramp, while the free flow after the ramp will increase. The boundary of the free flow region can be revealed by the free flow condition $\alpha'<\beta$. Thus we obtain

$$\gamma < \frac{\beta-\alpha}{1+\alpha}. \tag{14}$$

It is interesting to notice that the on-ramp becomes ineffective when γ assumes a value in between the above two regimes,

$$\frac{\beta-\alpha}{1+\alpha} < \gamma < \frac{\beta-\alpha}{\beta(1+\alpha)}. \tag{15}$$

Then we have

$$\rho_1 = \frac{\alpha}{1+\alpha} = j_1, \tag{16}$$

$$\rho_2 = \frac{1}{1+\beta} = 1-j_2, \tag{17}$$

where the free flow is present in first part of the lattice and the congestion is present in the second part of the lattice. As γ varies, both ρ_1 and ρ_2 are kept constant. The injection from the on-ramp only influences a transition layer around the on-ramp. The two parameters α' and β' can also be determined analytically,

$$\alpha' = \frac{\gamma(1+\alpha)\beta}{\beta-\alpha}, \tag{18}$$

$$\beta' = \frac{\gamma(1+\beta)\alpha}{\gamma(1+\alpha)(1+\beta)+\alpha-\beta}, \tag{19}$$

which are irrelevant to the bulk properties.

In summary, we observed three distinct phases as the three parameters α,β, and γ vary,

$$\text{(jam-jam) } \alpha>\beta, \quad \text{all } \gamma; \tag{20}$$

$$\text{(jam-jam) } \alpha<\beta, \quad \gamma>\frac{\beta-\alpha}{\beta(1+\alpha)}; \tag{21}$$

$$\text{(free-jam) } \alpha<\beta, \quad \frac{\beta-\alpha}{1+\alpha}<\gamma<\frac{\beta-\alpha}{\beta(1+\alpha)}; \tag{22}$$

$$\text{(free-free) } \alpha<\beta, \quad \gamma<\frac{\beta-\alpha}{1+\alpha}. \tag{23}$$

The phase diagram is shown in Fig. 1. As each section of the roadway can be characterized as free or jam, it is interesting to note that the jam-free phase is absent, i.e., the congestion will not be followed by the free flow. The numerical results can be correctly reproduced (see Fig. 2). As γ increases, the two transitions are obvious. The three phases can be easily discerned.

III. OFF-RAMP MODEL

Now we replace the on-ramp by an off-ramp. A particle on the ramp is removed with a rate γ. With similar considerations, we also obtain three distinct phases as the following (see Fig. 3):

$$\text{(free-free) } \alpha<\beta, \quad \text{all } \gamma; \tag{24}$$

$$\text{(free-free) } \alpha>\beta, \quad \gamma>\frac{\alpha-\beta}{\alpha(1+\beta)}; \tag{25}$$

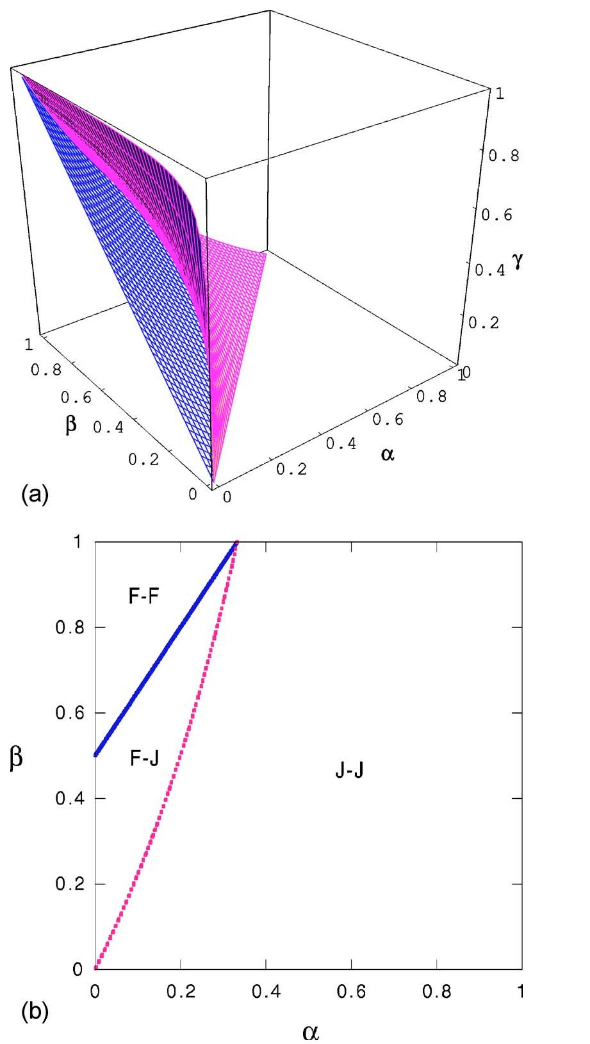

FIG. 1. (Color online) (a) Phase diagram on the parameter space (α,β,γ). A single on-ramp is presented. The black mesh marks the boundary between free-free phase and free-jam phase; the gray mesh marks the boundary between jam-jam phase and free-jam phase. The jam-jam phase is dominant. (b) 2D phase diagram on (α,β) at γ=0.5.

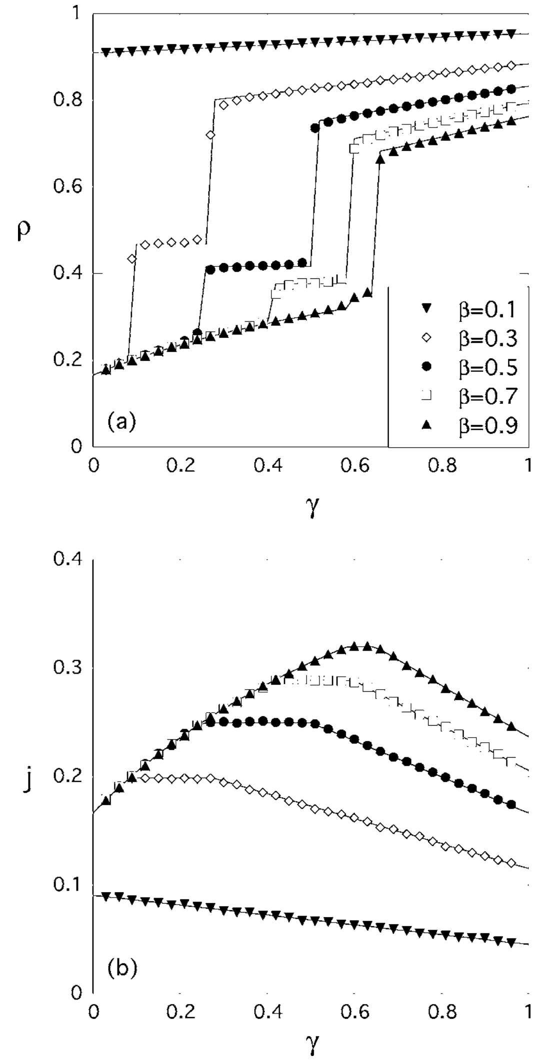

FIG. 2. On-ramp dependence of the bulk properties: (a) density and (b) current. The injection from the left boundary is fixed at α =0.2; the removal from the right boundary assumes five different values. Numerical results are shown by the symbols, which are obtained on a lattice of 1000 sites with a single on-ramp placed in the middle. Thus, $\rho=(\rho_1+\rho_2)/2$ and $j=(j_1+j_2)/2$. Data are obtained by average over 10^5 time steps. Analytic results are shown by the solid lines, which describe the data exactly. The free-jam phase presents a plateau in both figures.

$$\text{(free-jam) } \alpha > \beta, \quad \frac{\alpha-\beta}{1+\beta} < \gamma < \frac{\alpha-\beta}{\alpha(1+\beta)}; \tag{26}$$

$$\text{(jam-jam) } \alpha > \beta, \quad \gamma < \frac{\alpha-\beta}{1+\beta}. \tag{27}$$

In the free flow regime, the bulk properties are independent of β,

$$\rho_1 = \frac{\alpha}{1+\alpha} = j_1, \tag{28}$$

$$\rho_2 = \frac{\alpha(1-\gamma)}{1+\alpha} = j_2. \tag{29}$$

As γ increases, the bulk density after the ramp decreases, while the bulk density before the ramp holds constant. In the congestion regime, the bulk properties are independent of α,

$$\rho_1 = \frac{1}{(1+\beta)(1+\gamma)} = 1 - j_1, \tag{30}$$

$$\rho_2 = \frac{1}{1+\beta} = 1 - j_2. \tag{31}$$

As γ increases, the bulk density before the ramp decreases, while the bulk density after the ramp holds constant. In between these two regimes, we have

$$\rho_1 = \frac{\alpha}{1+\alpha} = j_1, \tag{32}$$

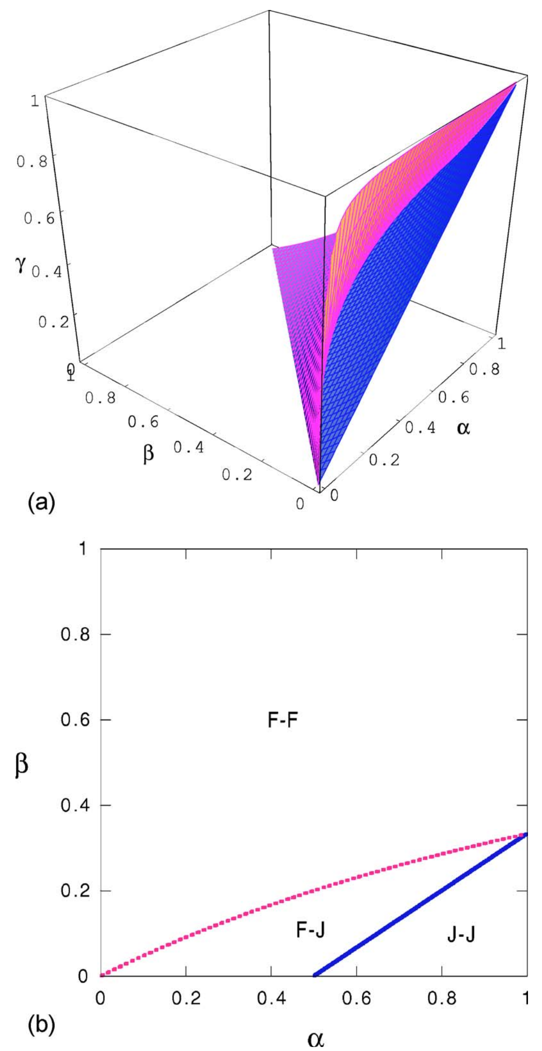

FIG. 3. (Color online) (a) Phase diagram on the parameter space (α,β,γ). A single off-ramp is presented. The black mesh marks the boundary between jam-jam phase and free-jam phase; the gray mesh marks the boundary between free-free phase and free-jam phase. The free-free phase is dominant. (b) 2D phase diagram on (α,β) at γ=0.5.

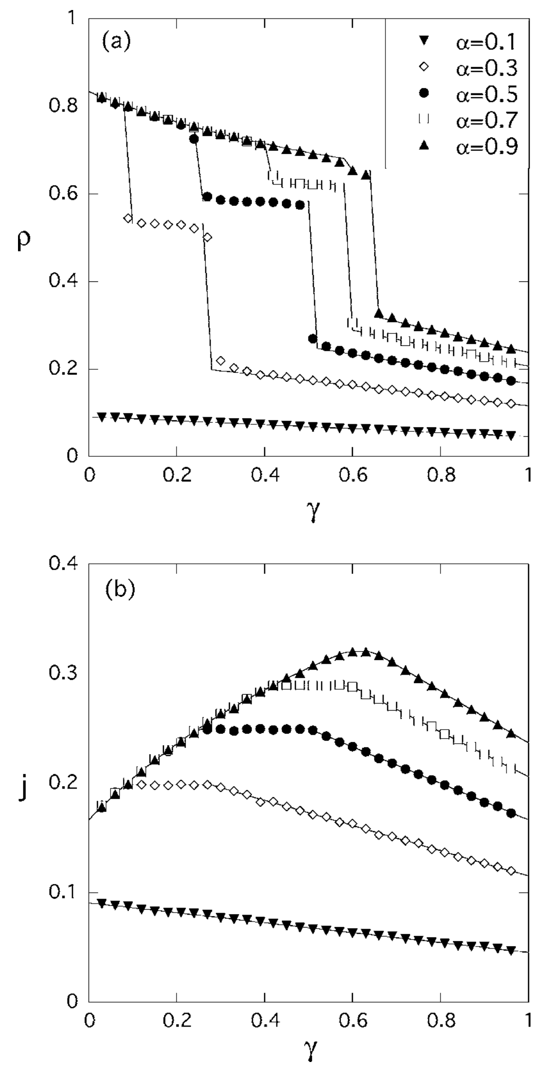

FIG. 4. Off-ramp dependence of the bulk properties: (a) density and (b) current. The injection from the left boundary assumes five different values; the removal from the right boundary is fixed at β=0.2. Numerical results are shown by the symbols; analytic results are shown by the solid lines. The results can be derived from Fig. 2 by the particle-hole symmetry.

$$\rho_2 = \frac{1}{1+\beta} = 1 - j_2. \tag{33}$$

The bulk properties are independent of γ. The free flow is present before the ramp, while the congestion is present after the ramp. It is interesting to note that the jam-free phase is still absent. The numerical results shown in Ref. [25] can be well reproduced (see Fig. 4).

As particles hopping forward can be reinterpreted as holes (empty sites) hopping backward, the cases of the off-ramp can be mapped into the cases of on-ramp: $\alpha \leftrightarrow \beta, \rho \leftrightarrow (1-\rho)$, (free flow) $\leftrightarrow$ (congestion).

IV. EFFECTS OF TWO RAMPS

The above results of a single ramp can be further extended to the roadway with many ramps. As the bulk properties are controlled by the boundaries, the roadway with many ramps can be taken as various homogeneous sections. Each section is specified by two effective rates, i.e., injection from the left and removal from the right. These effective rates can be solved analytically by the flow balance equations. First, we consider the case with an off-ramp followed by an on-ramp. To simplify the parametrization, we assume that the two ramps are operated by the same rate γ. The injection from the far left boundary and the removal from the far right boundary are still denoted by α and β, respectively. The roadway is now divided into three sections and each one can be either free or congested. With naive thinking, there are eight different phases in total. From the above results, however, the free flow will not follow the congestion. No matter if it is an on-ramp or an off-ramp, the jam-free phase is absent. Thus the number of distinct phases should be reduced to four: free-free-free, free-free-jam, free-jam-jam,

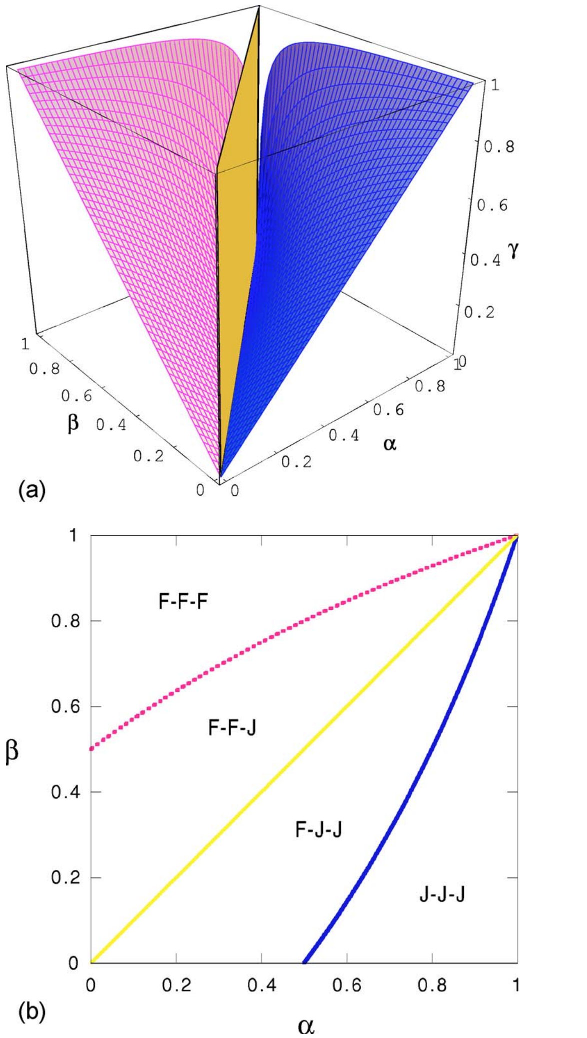

FIG. 5. (Color online) (a) Phase diagram on the parameter space (α,β,γ). Two ramps are presented. Along the traffic direction, the off-ramp comes before the on-ramp. The black mesh separates jam-jam-jam phase and free-jam-jam phase; the gray mesh separates free-free-free phase and free-free-jam phase. The boundary of $\alpha=\beta$ is also shown. (b) 2D phase diagram on (α,β) at $\gamma=0.5$.

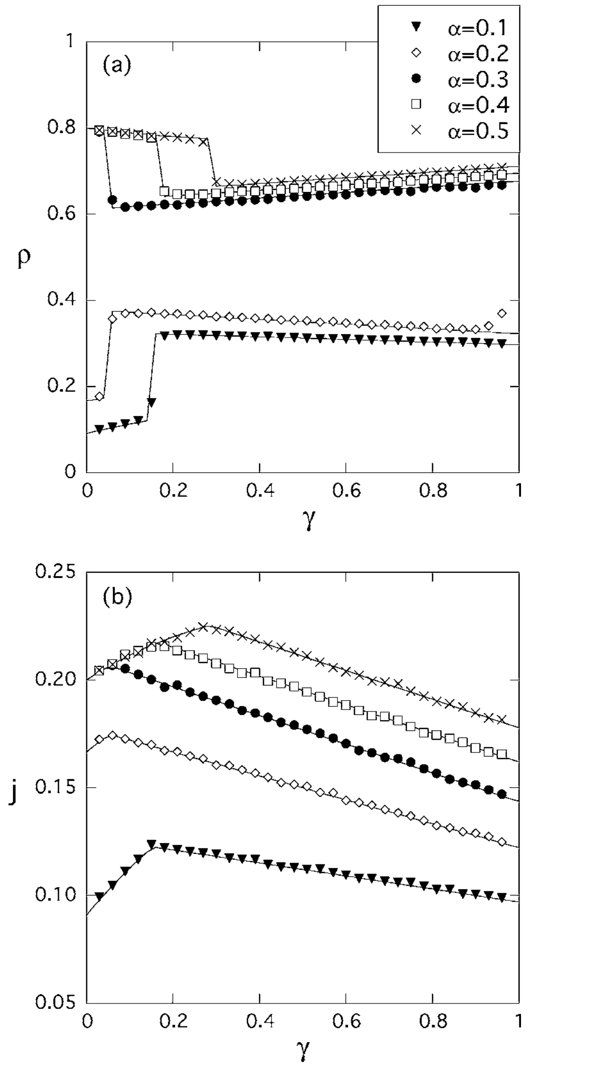

FIG. 6. Ramp dependence of the bulk properties: (a) density and (b) current. The injection from the left boundary assumes five different values; the removal from the right boundary is fixed at $\beta=0.25$. Numerical results are shown by the symbols, which are obtained on a lattice of 1500 sites with the off-ramp located at the 500th site and the on-ramp at the 1000th site. Thus, $\rho=(\rho_1+\rho_2+\rho_3)/3$ and $j=(j_1+j_2+j_3)/3$. Analytic results are shown by the solid lines, which describe the data exactly. Only one transition is observed.

and jam-jam-jam. With similar considerations, the regimes of these phases can be obtained as follows (see Fig. 5):

$$\text{(free-free-free)}\ \alpha<\beta,\quad \gamma<\frac{\beta-\alpha}{1-\alpha\beta}; \tag{34}$$

$$\text{(free-free-jam)}\ \alpha<\beta,\quad \gamma>\frac{\beta-\alpha}{1-\alpha\beta}; \tag{35}$$

$$\text{(free-jam-jam)}\ \alpha>\beta,\quad \gamma>\frac{\alpha-\beta}{1-\alpha\beta}; \tag{36}$$

$$\text{(jam-jam-jam)}\ \alpha>\beta,\quad \gamma<\frac{\alpha-\beta}{1-\alpha\beta}. \tag{37}$$

In the cases of free flow ($\alpha<\beta$), the last section of the roadway will be congested if γ is large enough; in the cases of congestion ($\alpha>\beta$), the first section of the roadway can be free of traffic jams if γ is large enough. The numerical results can be well reproduced (see Fig. 6). In contrast to the cases of a single ramp, there is only one transition when γ increases. The four phases can be further categorized into two groups. The variation of γ only leads to the transition within the group. The γ dependence of the global density will no longer be monotonic. In the cases of free flow, the density increases with the increase of γ before the transition; after the transition, the density decreases with the increase of γ. In the cases of congestion, the trend reverses as expected.

The situations would be much more complicated if the locations of the two ramps are switched, i.e., the on-ramp comes first and the off-ramp second. The phase diagram is

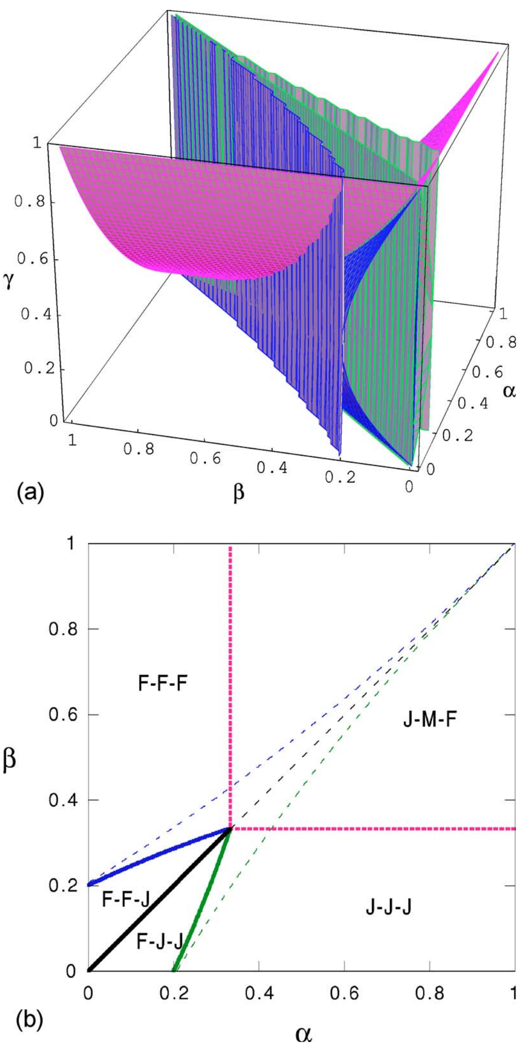

FIG. 7. (Color online) (a) Phase diagram on the parameter space (α,β,γ). Compared to Fig. 5, the locations of the two ramps are switched: the on-ramp comes before the off-ramp. The gray mesh marks the boundary of jam-maximum-free phase. The black mesh separates free-free-free phase and free-free-jam phase, which is also projected vertically to show the narrow strip $(\alpha\sim\beta)$ of the free-free-jam phase. The symmetry reflected on the boundary of $\alpha=\beta$ is obvious. (b) 2D phase diagram on (α,β) at $\gamma=0.5$.

shown in Fig. 7. The free-free-jam phase is now restricted within the following regime,

$$\alpha<\beta, \tag{38}$$

$$(1-\alpha\beta)^2+4(1+\alpha)(1+\beta)(\alpha-\beta)>0, \tag{39}$$

$$\mathrm{Max}\left(\frac{\beta-\alpha}{1+\alpha},x_-\right)<\gamma<x_+, \tag{40}$$

where

$$x_\pm=\frac{(1-\alpha\beta)\pm\sqrt{(1-\alpha\beta)^2+4(1+\alpha)(1+\beta)(\alpha-\beta)}}{2(1+\alpha)(1+\beta)}. \tag{41}$$

The free-free-jam phase is now sandwiched by the free-free-free phase. As γ increases monotonically, there are two-way transitions between these two phases.

Similarly, the free-jam-jam phase is restricted within the following regime,

$$\alpha>\beta, \tag{42}$$

$$(1-\alpha\beta)^2-4(1+\alpha)(1+\beta)(\alpha-\beta)>0, \tag{43}$$

$$\mathrm{Max}\left(\frac{\alpha-\beta}{1+\beta},y_-\right)<\gamma<y_+, \tag{44}$$

where

$$y_\pm=\frac{(1-\alpha\beta)\pm\sqrt{(1-\alpha\beta)^2-4(1+\alpha)(1+\beta)(\alpha-\beta)}}{2(1+\alpha)(1+\beta)}. \tag{45}$$

Again, the free-jam-jam phase is now sandwiched by the jam-jam-jam phase.

Furthermore, a new phase is observed when γ is large enough, which can be denoted as the jam-maximum-free phase. The section between the two ramps is saturated with the maximum current. The congestion emerges in the first section, while the free flow can still be maintained in the third section. We note that such a bottleneck effect is absent in the cases with a single ramp. The bulk properties can be obtained as follows:

$$\rho_1=\frac{1+\gamma}{2}=1-j_1, \tag{46}$$

$$\rho_2=\frac{1}{2}=j_2, \tag{47}$$

$$\rho_3=\frac{1-\gamma}{2}=j_3, \tag{48}$$

where the subscripts indicate the section of the roadway. The regime of such a phase can also be obtained,

$$\alpha<\beta,\quad \gamma>\frac{1-\alpha}{1+\alpha}; \tag{49}$$

$$\alpha>\beta,\quad \gamma>\frac{1-\beta}{1+\beta}. \tag{50}$$

The numerical results can be correctly described (see Fig. 8). As γ increases, three transitions can be observed. In the cases of congestion $(\alpha>\beta)$, the free flow is restored to the first section of the roadway as γ increases. However, as γ further increases, the free flow disappears and the congestion dominates the entire roadway once again. When γ continues to increase, the middle section of the roadway saturates and

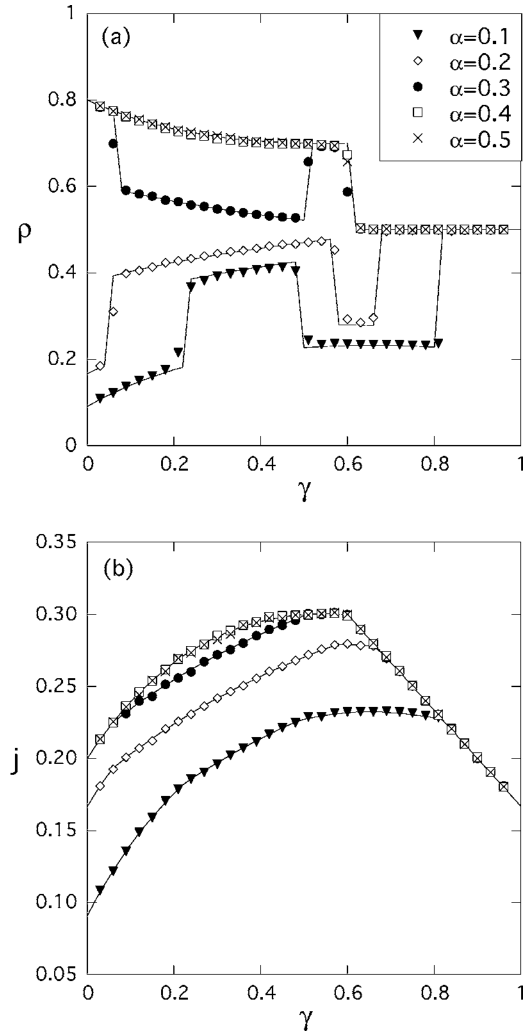

FIG. 8. Ramp dependence of the bulk properties: (a) density and (b) current. The injection from the left boundary assumes five different values; the removal from the right boundary is fixed at $\beta=0.25$. Numerical results are shown by the symbols, which are obtained on a lattice of 1500 sites with the on-ramp located at the 500th site and the off-ramp at the 1000th site. Analytic results are shown by the solid lines, which describe the data exactly. Three transitions can be observed. The distributions remain the same for $\alpha>0.37$.

becomes the bottleneck of the traffic. Thus the system starts with the jam-jam-jam phase, changes into the free-jam-jam phase, returns to the jam-jam-jam phase, and finally saturates to the jam-maximum-free phase. When the difference between α and β is large enough, the free-jam-jam phase becomes impossible, i.e.,

$$(1-\alpha\beta)^2-4(1+\alpha)(1+\beta)(\alpha-\beta)<0. \tag{51}$$

The jam-jam-jam phase will then transit to jam-maximum-free phase directly. In such cases, a scaling relation is observed. The bulk properties remain unchanged as the difference between α and β further increases. Similar results can also be observed in the cases of free flow ($\alpha<\beta$).

V. CONCLUDING REMARKS

In this paper, we study the effects of ramps in the asymmetric simple exclusion processes. The system can be divided into homogeneous sections connected by the ramps. We show that the ramp flow can be replaced by two effective rates of injection and removal. The values of these effective rates can be obtained by balancing the flow. The complete classification of distinct phases can be achieved. The exact phase diagrams in the parameter space (α,β,γ) are obtained analytically.

With a single ramp (on or off), three different phases are observed. The phase regime depends on the three parameters: α,β, and γ. In each phase, however, the bulk properties depend only on two of the three parameters. For example, in the free-jam phase, γ has to be bounded to the limits determined by α and β. Yet the bulk properties are independent of γ. Before the ramp, both the density and the flow are controlled by α alone; after the ramp, the properties are determined by β alone. Thus, for the jam-jam phase, α is irrelevant to the bulk properties; for the free-free phase, β is irrelevant; and for the free-jam phase, γ is irrelevant. The jam-free phase is forbidden, i.e., the conventional bottleneck cannot be realized by a single ramp. As the ramp flow increases, two transitions can be observed. The absence of the jam-free phase can be attributed to the particle-hole symmetry of ASEP. For highway traffic, the acceleration is much stronger than the deceleration. The so-called extended hopping will break the symmetry and shift the maximum flow toward the low density region. When the symmetry is violated, the conventional bottleneck effect is expected to emerge.

When both an on-ramp and an off-ramp are presented, the phase diagram depends strongly on the order of the two ramps. When the off-ramp comes before the on-ramp along the traffic direction, four different phases are observed and there is only one transition as the ramp flow increases. When the off-ramp comes after the on-ramp, one more phase is realized and three transitions can be observed as the ramp flow increases. In this new phase, the flow saturates in between the two ramps, which becomes the bottleneck to the traffic flow. The bulk properties are solely determined by the ramp flow. To fully explore the cases of two ramps, one more parameter has to be introduced, i.e., the two ramps are operated independently. The parameter space becomes four-dimensional. Exact results can be obtained analytically by the same approach.

With three or more ramps, the analytical results can also be reached straightforwardly. In this simple model, the dynamics in bulk is deterministic and short ranged. The phase transitions are triggered by the stochastic boundaries. It would be interesting to further study the influences of stochasticity and/or long-range interactions in bulk, which would provide the next step toward an analytical description of the real traffic.

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